

# A TYPE B ANALOGUE TO RIBBON TABLEAUX

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**ABSTRACT.** We introduce a shifted analogue of the ribbon tableaux defined by James and Kerber [3]. For any positive integer  $k$ , we give a bijection between the  $k$ -ribbon fillings of a shifted shape and regular fillings of a  $\lfloor k/2 \rfloor$ -tuple of shapes called its  $k$ -quotient. We also define the corresponding generating functions, and prove that they are symmetric, Schur positive and Schur  $Q$ -positive.

## 1. INTRODUCTION

The study of ribbon tableaux on shifted shapes combines two existing areas of work: the theory of ribbon tableaux and Schur's  $Q$ -functions. Ribbon tableaux introduced by James and Kerber[3] have applications to the representations of the symmetric group over a field of finite characteristic. Their theory was extended to the LLT polynomials by Lascoux, Leclerc and Thibon which arise in the Fock space representation of the universal enveloping algebra of quantum affine  $\mathfrak{sl}_n$ [5]. An expansion of Macdonald polynomials in terms of LLT polynomials is given in [2], which combined with the Schur positivity result for LLT polynomials[6] provided a combinatorial proof of the Macdonald positivity conjecture.

Schur's  $Q$ -functions come up as the symmetric functions that correspond to the shifted diagrams. They have a connection to the irreducible spin characters of the symmetric group, analogous to that of Schur functions and irreducible characters of linear representations[7]. Since their introduction in [10], applications to diverse mathematical fields have been discovered, including the cohomology of isotropic Grassmannians [4] and polynomial solutions to the BKP equation in hydrodynamics.

In this work, we are merging these two ideas to initiate a theory of ribbon tableaux for shifted shapes. The  $k$ -quotients and  $k$ -cores for shifted shapes were previously studied by Morris and Yaseen in 1986[8]. We expand upon their work, reformulating it in a more explicit way that is analogous to the ribbon tilings of unshifted shapes due to James and Kerber[3]. We also look at standard and semi-standard fillings of these shapes, and define shifted  $k$ -ribbon functions. We give a positive expansion in terms of Schur's  $Q$ -functions, analogous to the unshifted case.

The positivity result hints at the possibility of defining a type B analogue for the LLT polynomials. We show that there is no natural expansion of the spin statistic to the shifted ribbons, and provide some counter examples that should prove valuable for further research.

The layout of this paper is as follows: In Section 2, we recall the notions of Schur functions, Schur's  $Q$ -functions and ribbon tableaux. In Section 3, we give a graphical description of  $k$ -ribbons on a shifted diagram, which differs from the standard case in that we have some 'double ribbons', which are allowed to contain  $2 \times 2$  boxes. We define the shifted  $k$ -ribbon tableaux and the corresponding  $P$  and  $Q$  functions, as well as state our main theorem giving an expansion of a shifted ribbon  $Q$  function in terms of Schur  $Q$  functions. Sections 4 and 5 give the combinatorial constructions necessary to prove this, including a new type of object that comes up in shifted  $k$ -quotients which we call folded tableaux. We give bijections between ribbon fillings of a shifted diagram and its  $k$ -quotient, both in the standard and semi-standard case. In Section 5, we give a description of the shifted ribbon functions in terms of peak functions. Lastly, in Section 7, we discuss the difficulties of defining a natural analogue of the spin statistic from the unshifted case, and provide some counter-examples.

## 2. PRELIMINARIES

**2.1. Schur Functions.** A *partition* of  $n$  is a list  $\mu = (\mu_1, \mu_2, \dots, \mu_k)$  of non-increasing positive integers adding up to  $n$ , called the parts. Here,  $n$  is called the *size* of the partition, denoted  $|\mu|$ , and the number of its parts is called its *height* denoted  $\text{ht}(\mu)$ . With every partition, we associate a *Young diagram*, an array with  $\mu_i$  boxes on row  $i$ .

A *semi-standard Young tableau* of shape  $\mu$  is a filling of its boxes with positive integers such that each column will be increasing from bottom to top, and each row will be non-decreasing from left to right. A semi-standard tableau that contains each of the numbers from 1 to  $n$  exactly once is called *standard*. We will denote the set of semi-standard tableaux of shape  $\mu$  by  $SSYT(\mu)$ , and the set of standard ones by  $SYT(\mu)$ .

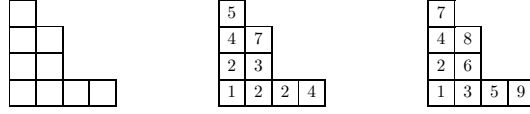


FIGURE 1. The diagram  $\mu = (4, 2, 2, 1)$  with corresponding semi-standard and standard fillings.

For a partition  $\mu$  we define its *Schur function* as follows:

$$(1) \quad s_\mu(X) = \sum_{T \in SST(\mu)} X^{|T|}$$

Here  $X^{|T|}$  is the monomial where the power of  $x_i$  is equal to the number of times  $i$  occurs in  $T$ . The semi-standard filling in Figure 1 corresponds to the monomial  $x_1 x_2^3 x_3 x_4^2 x_5 x_7$ .

The *reading word* of a tableau is a reading of all its labels from left to right, top to bottom. For example, the semi-standard tableau from Figure 1 has the reading word 547231224, where as the standard one has the reading word 748261359.

Note that the reading word of a standard tableau  $S$  gives a permutation of numbers from 1 to  $|n|$ , so we can talk about its descent, peak and spike sets. The *descent set* of a standard tableau  $T$  is defined as follows:

$$\text{Des}(T) = \{i \mid i \text{ is to the left of } i+1 \text{ in the reading word of } T\} \subset [n-1]$$

In general, for any set  $D \subset [n]$ , the peak and spike sets of  $D$  are given by:

$$\begin{aligned} \text{Peak}(D) &= \{i \mid i \in D \text{ and } i-1 \notin D\} \\ \text{Spike}(D) &= \{i \mid i \in D \text{ and } i-1 \notin D \text{ or } i \notin D \text{ and } i-1 \in D\}. \end{aligned}$$

Throughout this work, we will mainly be interested in the case when  $D$  is the descent set of the reading word for a tableau. For a tableau  $T$ , we will use the notations  $\text{Peak}(T)$  and  $\text{Spike}(T)$  to denote  $\text{Peak}(\text{Des}(T))$  and  $\text{Spike}(\text{Des}(T))$  respectively.

The standard tableau from Figure 1 has descent, peak and spike sets  $\{1, 3, 5, 6\}$ ,  $\{3, 5\}$  and  $\{2, 3, 4, 5, 7\}$  respectively. In 1984, Gessel[1] has shown that the Schur function for a partition  $\mu$  can be expressed in terms of descent sets:

$$s_\mu(X) = \sum_{T \in SYT(\mu)} F_{\text{Des}(T)}(X)$$

where  $F_D(X)$ ,  $D \in [n-1]$  denotes Gessel's fundamental basis for quasisymmetric functions defined by:

$$(2) \quad F_D(X) = \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_m \\ t \in D \Rightarrow i_t \neq i_{t+1}}} x_{i_1} x_{i_2} x_{i_3} \dots x_{i_m}$$

This formula allows us to calculate the Schur function of a partition using only its standard fillings. For example, the Schur function of  $(3, 2)$ , whose standard fillings are given in Figure 2 is:

$$s_{(3,2)}(X) = F_3(X) + F_2(X) + F_4(X) + 2F_{2,4}(X)$$

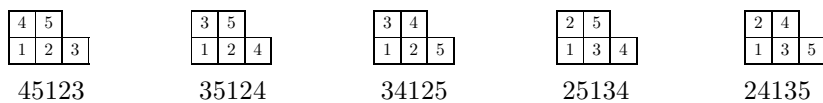
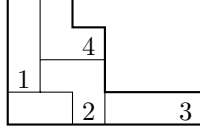


FIGURE 2. The standard tableaux of shape  $(3, 2)$  and their reading words

FIGURE 3. A 3-ribbon tableau of shape  $\mu = (6, 3, 3, 2)$ 

**2.2. Ribbon Tableaux.** A  $k$ -ribbon on an unshifted diagram is a connected skew diagram that contains no  $2 \times 2$  square. A  $k$ -ribbon  $R$  is removable from diagram  $\mu$  if  $\mu \setminus \nu = R$  for some  $\nu \subset \mu$ . A diagram with no removable  $k$ -ribbon is called a  $k$ -core.

On an unshifted shape, we will describe the diagonal of a cell  $C$  by the difference of its row index and column index, with the convention that the cell  $C_0$  in the bottom left hand corner has  $\text{rowindex}(C_0) = \text{colindex}(C_0) = 1$  and  $\text{diag}(C_0) = \text{rowindex}(C_0) - \text{colindex}(C_0) = 0$ . For a given  $k$ -ribbon  $R$ , we will call the cell with the highest diagonal the *head* of the ribbon. A set of disjoint ribbons form a *horizontal strip* if their disjoint union is a (not necessarily connected) skew-shape and their heads lie on different columns. A *semi-standard  $k$ -ribbon tableau* of shape  $\mu$  is a sequence of shifted diagrams  $\mu_0 \subset \mu_1 \subset \dots \subset \mu_n = \mu$  where  $\mu_0$  is a  $k$ -core, and each  $\mu_i \setminus \mu_{i-1}$  is a horizontal  $k$ -ribbon strip, the ribbon on which we label by  $i$ . The generating function for the  $k$ -ribbon tableaux of shape  $\mu$  is given by:

$$GF_{\mu}^{(k)}(X) = \sum_{SSRT_k(\mu)} X^T$$

where  $SSRT_k(\mu)$  denotes the set of semi-standard  $k$ -ribbon tableaux of shape  $\mu$  and  $X^T$  is the monomial where the power of  $x_i$  is given by the number of times  $i$  occurs in  $T$ .

James and Kerber [3] showed that there is a weight-preserving bijection between semi-standard ribbon tableaux of shape  $\mu$ , and semi-standard fillings of a  $k$ -tuple of unshifted shapes  $(\mu^0, \mu^1, \dots, \mu^{k-1})$  called the  *$k$ -quotient* of  $\mu$ . This shows that:

$$GF_{\mu}^{(k)}(X) = s_{\mu^0}(X) s_{\mu^1}(X) \dots s_{\mu^{k-1}}(X)$$

The *spin* of a ribbon  $R$ , defined by Lascoux, Leclerc and Thibon [5] is  $(|R| - \text{ht}(R) - 1)/2$ , which is not an integer for some cases. For a semi-standard  $k$ -ribbon tableaux  $T$  of shape  $\mu$ , we define the *spin* of  $T$  to be the sum of the spins of all ribbons on  $T$ . The *cospin* of  $T$  is given by  $\text{spin}(T^*) - \text{spin}(T)$  where  $T^*$  is the semi-standard  $k$ -ribbon tableaux of shape  $\mu$  with the maximum spin. The cospin is an integer for every tiling  $T$ .

Multiplying each tableau by a variable  $q$  raised to its cospin gives us the LLT-polynomial, which can be written as a sum of schur polynomials with coefficients from  $\mathbb{Z}^+[q]$ :

$$G_{\mu}(X; q) = \sum_T X^T q^{\text{cospin}(T)}$$

**2.3. Shifted Tableaux.** A partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$  is called *strict* if all its parts are distinct. With every strict partition, we associate a *shifted diagram*, which is an array with  $\lambda_i$  boxes on row  $i$ , where row  $i$  is shifted  $k - i$  steps to the right, forming a staircase shape.

A *semi-standard shifted tableau* of shape  $\lambda$  is a filling of its boxes with elements from the marked alphabet  $1' < 1 < 2' < 2 < 3' < 3 \dots$  such that each row will be non-decreasing from left to right with no repeated marked numbers, and each column will be non-decreasing from bottom to top with no repeated unmarked numbers. A semi-standard shifted tableau of shape  $\lambda$  that contains each of the numbers  $1, 2, \dots, |\lambda|$  exactly once possibly marked, it is called *marked standard*, and if they are all unmarked it is called *standard*. We will denote the set of semi-standard shifted tableaux of shape  $\lambda$  by  $SSShT(\lambda)$ , the set of marked standard ones by  $SShT \pm(\lambda)$  and the set of the standard ones by  $SShT(\lambda)$ .



FIGURE 4. The shifted diagram for  $\lambda = (4, 3, 1)$  with a corresponding semi-standard, marked standard and standard filling.

The Schur  $Q$  and  $P$  functions for a strict partition  $\lambda$  are defined as follows:

$$(3) \quad Q_\lambda(X) = \sum_{S \in SSShT(\lambda)} X^{|S|}$$

$$(4) \quad P_\lambda(X) = 2^{-ht(\lambda)} \sum_{S \in SSShT(\lambda)} X^{|S|} = \sum_{S \in SSShT^*(\lambda)} X^{|S|}$$

where  $S \in SSShT^*(\lambda)$  denotes the set of semi-standard tableaux of shape  $\lambda$  with no marked entries on the main diagonal, and  $X^{|S|}$  is the monomial where the power of  $x_i$  is equal to the number of times  $i$  or  $i'$  occurs in  $S$ . The semistandard filling in Figure 4, for example, corresponds to the monomial  $x_1^2 x_2^3 x_3^2 x_4$ .

The *reading word* of a shifted tableau is, like in the unshifted case, a reading of all its labels from left to right, top to bottom. The definitions of descent, peak and spike sets can be extended to the reading words of marked standard tableaux by reflecting the marked coordinates to the beginning and working with the corresponding word, so that  $\text{Des}(74'6'812'3'5) = \text{Des}(32647815) = \{1, 2, 5\}$ .

**Lemma 2.1.** *If  $i \in \text{Des}(T)$ , then  $i \in \text{Des}(T')$  if and only if  $i$  is unmarked in  $T'$ . If  $i \notin \text{Des}(T)$ , then  $i \in \text{Des}(T')$  if and only if  $i + 1$  is marked in  $T$ .*

*Proof.* This follows directly from the definition of descents on marked tableaux.  $\square$

Like in the case of Schur functions, Schur's  $Q$ -functions can be expanded in terms of the fundamental quasisymmetric functions:

$$Q_\lambda(X) = \sum_{T' \in SShT_\pm(\lambda)} F_{\text{Des}(T')}(X)$$

For this expansion, we only look at the marked standard tableaux of shape  $\lambda$ . An expansion that also eliminates the markings and only considers the standard fillings was given by Stembridge[12]:

$$Q_\lambda(X) = \sum_{T \in SShT(\lambda)} 2^{|\text{Peak}(T)|+1} G_{\text{Peak}(T)}(X)$$

Here, the functions  $G_P$ , where  $P$  is a subset of  $[2, 3, \dots, n-1]$  with no consecutive entries, are the *peak functions* are defined in [12] by:

$$G_P(X) = \sum_{\substack{D \in [n-1] \\ \text{Spike}(D) \supset P}} F_D(X)$$

Given two strict partitions  $\mu$  and  $\lambda$  with  $\mu \subset \lambda$ , the *skew-shifted diagram*  $\lambda \setminus \mu$  is the diagram for  $\lambda$  with the cells corresponding to the diagram of  $\mu$  deleted.



FIGURE 5. The skew-diagram  $(4, 3, 1) \setminus (3, 1)$  with a corresponding semi-standard, marked standard and standard filling.

We can apply the above definitions to the skew shifted diagrams to get skew shifted tableaux. More precisely, the set of semi-standard shifted tableaux of shape  $\lambda \setminus \mu$ , denoted  $SSShT(\lambda \setminus \mu)$  is given by all the fillings of  $\lambda \setminus \mu$  from the marked alphabet with non-decreasing columns and rows such that we have no unmarked numbers repeated along columns and no marked numbers repeated along rows. We will denote the marked standard fillings of  $(\lambda \setminus \mu)$  (where we use each number from 1 to  $n$  once, possibly marked)

by  $SShT \pm (\lambda \setminus \mu)$  and its standard fillings (where we use each number from 1 to  $n$  once, unmarked) by  $SShT(\lambda \setminus \mu)$ . This gives rise to a skew analogue for Schur's  $Q$ -function:

$$(5) \quad Q_{\lambda \setminus \mu}(X) = \sum_{S \in SShT(\lambda \setminus \mu)} X^{|S|} = \sum_{T' \in SShT \pm (\lambda \setminus \mu)} F_{Des(T')}(X) = \sum_{T \in SShT(\lambda \setminus \mu)} G_{Peak(T)}(X)$$

It was shown by Stembridge that the skew shifted  $Q$ -functions expand positively into Schur's  $Q$ -functions:

**Theorem 2.2.** (Stembridge [11]) *There exist coefficients  $f_{\mu, \nu}^\lambda \in \mathbb{N}$  satisfying:*

$$Q_{\lambda \setminus \mu}(X) = \sum_{\nu} f_{\mu, \nu}^\lambda Q_{\nu}(X) \quad Q_{\mu}(X)Q_{\nu}(X) = \sum_{\lambda} f_{\mu, \nu}^\lambda Q_{\lambda}(X)$$

where  $f_{\mu, \nu}^\lambda = 0$  unless  $|\mu| + |\nu| = |\lambda|$ .

### 3. SHIFTED RIBBON TABLEAUX

**3.1. Ribbons on Shifted Diagrams.** On a shifted diagram, we call the columns strictly to the left of the last row its *shifted region*, and the rest its *unshifted region*. Note that the unshifted region uniquely determines the diagram<sup>1</sup>.

The definition of the hook of a cell on a shifted diagram depends whether the cell falls into the shifted region. For any cell  $C$  in the unshifted region, the *hook* of  $C$  is the union of  $C$ , with the cells above it in its column, and the cells to its right in the row. For a cell in the shifted region, its hook additionally includes the row of cells directly above the highest cell in the column of  $C$ . The number of cells in its hook is called the *hook length* of  $C$ .

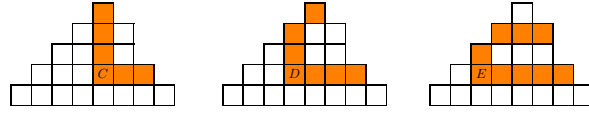


FIGURE 6. Hooks of the cells  $C$ ,  $D$  and  $E$

For a cell  $C$  on the shifted shape, we define its diagonal value to be  $diag(C) = row(C) - col(C) + 1$  counting from the bottom left corner (The cells  $C$ ,  $D$  and  $E$  in Figure 6 have diagonal values 4, 3 and 2 respectively). Because of its construction, all cells on a shifted diagram will have a diagonal value  $\geq 1$ , and we will call the diagonal labeled 1 the *main diagonal* of  $\lambda$ .

**Definition 3.1.** We define a single-ribbon on  $\lambda$  to be a connected skew-diagram with each cell on a different diagonal (Equivalently, not containing any  $2 \times 2$  square). A single ribbon  $R$  is removable if  $\lambda - R$  is also a shifted diagram.

Some important notations we will use about ribbons throughout the paper are heads and tails of ribbons. The cell with the highest diagonal value will be called the *head* of  $R$ , denoted by  $H(R)$ , and the one with the lowest will be called the *tail* of  $R$ , denoted  $T(R)$ . We will also use  $|R|$  for the size of a ribbon (the number of cells it contains) and  $ht(R)$  for the height of a ribbon (the number of rows of  $\lambda$  that it intersects).

Note that for a single-ribbon  $R$ ,  $|R| = diag(H_R) - diag(T_R) + 1$

**Definition 3.2.** A double-ribbon of size  $k$  is a union of two disjoint single-ribbons  $R$  and  $S$  of sizes  $r \geq s$  with  $r + s = k$  such that the tail of  $R$  is on the main diagonal of  $\lambda$ , and the tail of  $S$  is on the main diagonal of  $\lambda - R$ , and their union forms a skew-shifted shape. A double ribbon  $Q$  is removable if  $\lambda - Q$  is also a shifted diagram.

The head of  $Q = (R, S)$  is the head of  $R$ , and its tail is the tail of  $s$ .

**Definition 3.3.** A  $k$ -ribbon is a single or double ribbon of size  $k$ .

**Proposition 3.4.** For any removable  $k$ -ribbon  $R$  on  $\lambda$ , there is no cell on  $R$  strictly to the right or strictly below  $H(R)$ .

<sup>1</sup>We deviate from convention in defining the shifted region, to more easily define hook lengths.



FIGURE 7. A single ribbon(left) and a double ribbon (right)

*Proof.* As  $\lambda \setminus R$  is also a shifted diagram, if  $R$  includes a cell strictly to the right of  $H(R)$ , it will also contain the cell to the right of  $\lambda$  in the same row. Similarly, if  $R$  has a cell below  $H(R)$ , it will also include the cell below  $\lambda$  in the same column. In both cases,  $R$  has a cell with a higher diagonal value than  $H(R)$ , giving us a contradiction.  $\square$

**Proposition 3.5.** *A shifted shape  $\lambda$  has a removable single  $k$ -ribbon with  $\text{diag}(H_R) = m$  if and only if it has a part of size  $m + k$  and no part of size  $m$ . If exists, it is the unique ribbon  $R$  where  $\lambda - R$  has the part  $m + k$  replaced with  $m$ . Furthermore,  $\lambda$  has a removable double  $k$ -ribbon with  $\text{diag}(H_R) = a$  if and only if it two parts of size  $a$  and  $k - a < a$ , the ribbon being the unique  $R$  where  $\lambda - R$  is  $\lambda$  with the two parts removed.*

*Proof.* If  $\lambda$  has a part of size  $m + k$  and no part of size  $m$ , then removing the outermost box of diagonals  $m + k$  to  $m + 1$  gives us a ribbon of size  $k$ , with  $\text{diag}(H_R) = m$ . As the ribbon is removable,  $\lambda \setminus R$  is itself a skew-diagram, which means there is no cell on  $\lambda$  above  $T_R$  or to the left of  $H_R$ , and  $R$  contains the outermost cell of each diagonal from  $m + k$  to  $m + 1$ . As  $H_R$  is at the end of a row, and has diagonal value  $m + k$ ,  $\lambda$  contains a row of size  $m + k$ . Similarly,  $T_R$  is on a row with at least  $m + 1$  cells, and as the row above has no cell above  $T_R$ , it has less than  $m$  cells. The case for the double ribbon follows as the double ribbon is a union of two single ribbons, and a ribbon of size  $a$  with  $\text{diag}(H_R) = a$  will have its tail on the main diagonal.  $\square$

A corollary of this proposition is that no shape can have a double ribbon of size  $(t, t)$ .

**Theorem 3.6.** *A shifted diagram  $\lambda$  admits no removable  $k$ -ribbon iff it has no cells with hook length equal to  $k$ .*

In this case, we call  $\lambda$  a  $k$ -core.

*Proof.* We claim that there is a bijection between removable  $k$ -ribbons and cells with hook length equal to  $k$ , where cells in the unshifted part correspond to single ribbons and cells in the shifted part correspond to double ribbons. Under this bijection it is clear that if a diagram admits no  $k$  ribbons, it can not have a cell with hook length  $k$  and vice versa.

Let  $C$  have hook length equal to  $k$ . First let us look at the case when  $C$  is in the unshifted part. Let  $R$  be the single ribbon consisting of the outermost cell on each diagonal that the hook of  $C$  passes. This means, as the hook contains  $k$  cells,  $R$  has size  $k$ . Furthermore, the head and tail of  $R$  are the endpoints of the hook of  $C$ , so there is no cell above  $T(R)$  or to the left of  $H(R)$ , so  $R$  is removable. Conversely, if  $R$  is a single  $k$ -ribbon, the cell on the row of  $H(R)$  and the column of  $T(R)$  has hook length  $k$  and is on the unshifted part.

Now let us assume  $C$  is a cell in the shifted part, so that its hook includes the row above the column of  $C$ . Assume this row has length  $t$ . This means,  $C$  is on a row of size  $t - k$ , and by Proposition 3.4 the shape has a unique removable double ribbon of size  $(t, k - t)$ . Conversely, if  $R$  is a removable double ribbon of size  $(t, k - t)$  with  $t < t - k$ , the diagram has a rows  $i$  and  $j$  with sizes  $t$  and  $t - k$ . The cell on row  $j$  and column right below row  $i$  falls on the shifted part and has hook length  $k$ .  $\square$

**3.2. The  $k$ -Abacus Correspondence.** In this section, we will show that the ribbons on shifted tableaux can be expressed using the  $k$ -abacus notation of James and Kerber[3]. A  $k$ -abacus consists of runners labeled by  $1, 2, 3 \dots k$ , and numbers placed on these runners as follows:

1	2	3	...	k
1	2	3	...	k
k+1	k+2	k+3	...	2k
2k+1	2k+2	2k+3	...	3k
...	...	...	...	...

To any shifted diagram  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  we will associate the  $k$ -abacus with beads on positions  $\lambda_1, \lambda_2, \dots, \lambda_n$ . For example, for the diagram  $\lambda = (16, 11, 10, 9, 8, 7, 4, 3, 1)$  we get the 5-abacus:

$a_1$	$a_2$	$a_3$	$a_4$	$a_5$
●	○	●	●	○
○	●	●	●	●
●	○	○	○	○
●	○	○	○	○
○	○	○	○	○
...	...	...	...	...

Given a strict partition  $\lambda$ , we identify each runner  $a_i$  in its abacus with a shifted shape  $\alpha^{(i)}$ , by treating the runners as the abacus of a shifted 1-core. More precisely,  $\alpha^{(i)}$  will be the shifted shape  $(\alpha_1^{(i)}, \alpha_2^{(i)}, \dots, \alpha_t^{(i)})$  where  $k(\alpha_1^{(i)} - 1) + i, \dots, k(\alpha_t^{(i)} - 1) + i$  are the parts of  $\lambda$  that are equal to  $i$  modulo  $k$ . We will call the  $k$ -tuple  $(\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(n)})$  the abacus representation  $\alpha$ . The 5-abacus representation of the example  $(16, 11, 10, 9, 8, 7, 4, 3, 1)$  with the abacus above is  $(431, 2, 21, 21, 2)$ .

There are two types of moves allowed on the  $k$ -abacus of  $\lambda[9]$ :

- *Type I Move*: Sliding one bead one position higher in its runner, if that position is unoccupied, or removing a bead from the top row of column  $k$
- *Type II Move*: Removing two beads from the first row, if they are on two conjugate runners.

After a move on the  $k$ -abacus, we get a new shifted diagram  $\lambda^* \subset \lambda$ . A *Type I* move corresponds to replacing a part of size  $m + k$  with one of size  $m$ , whereas a *Type II* move corresponds to removing to parts of sizes adding up to  $k$ . By Proposition 3.5, we have the following correspondence:

**Corollary 3.7.** *Making a move on the  $k$ -abacus of  $\lambda$  is equivalent to removing a  $k$ -ribbon from  $\lambda$ . In particular,*

- (1) *Single-Ribbon Correspondence: Making the Type I move from position  $m + k$  to position  $m$  is equivalent to removing a single-ribbon  $A$  with  $\text{diag}(H_A) = m + k$  and  $\text{diag}(T_A) = m + 1$  (where  $m = 0$  is removing a bead from the top row of column  $k$ ).*
- (2) *Double-Ribbon Correspondence: Making the Type II move removing top beads  $t$  and  $k - t$  from conjugate runners  $a_t$  and  $a_{k-t}$  equivalent to removing a double-ribbon of size  $(t, k - t)$ .*

**Theorem 3.8.** *The  $k$ -core of a shifted diagram is unique.*

*Proof.* By the Corollary above, the  $k$ -core of a shifted diagram corresponds to the  $k$ -core of the abacus, and the final state of the abacus is in which is independent of the order of the moves [3]. □

### 3.3. Shifted Ribbon Tableaux.

**Definition 3.9.** *A standard  $k$ -ribbon tableau of shape  $\lambda$  is a sequence of shifted diagrams  $\lambda^{(0)} \subset \lambda^{(1)} \subset \dots \subset \lambda^{(n)} = \lambda$ , where each  $A_i = \lambda^{(i)} \setminus \lambda^{(i-1)}$ ,  $i = 1, 2, \dots, n$  is a  $k$ -ribbon, and  $\lambda^{(0)}$  is a  $k$ -core. We label the ribbon  $A_i$  by  $i$ .*

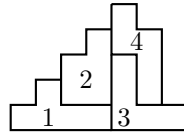


FIGURE 8. A 5-Ribbon Tableau with no 5-core of shape  $\lambda = (7, 5, 4, 3, 1)$

**Definition 3.10.** *A skew-diagram  $S$  on a shifted diagram  $\lambda$  is called a horizontal  $k$ -ribbon strip (resp. vertical  $k$ -ribbon strip) if there exists a sequence of shifted diagrams  $\lambda^{(0)} \subset \lambda^{(1)} \subset \dots \subset \lambda^{(t)} = \lambda$ , where:*

- (1) *Each  $R_i := \lambda^{(i)} \setminus \lambda^{(i-1)}$  is a  $k$ -ribbon.*
- (2)  *$H(R_i)$  is strictly to the right of (resp. strictly above)  $H(R_{i-1})$  for each  $i$ .*
- (3)  *$S = \bigcup_{i=1}^n R_i = \lambda \setminus \lambda^{(0)}$ .*

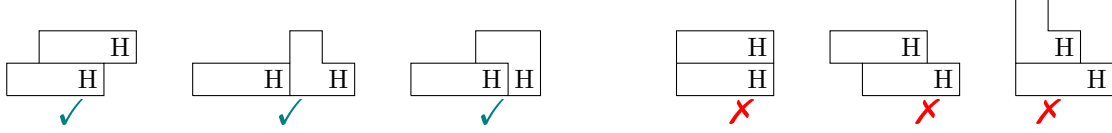
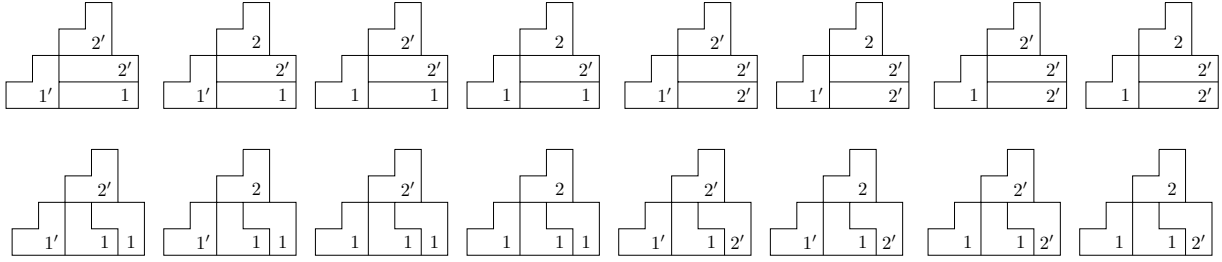


FIGURE 9. Horizontal strip examples and non-examples

**Definition 3.11.** A semi-standard  $k$ -ribbon tableau is given by a sequence  $\lambda^{(0)} \subset \lambda^{(1')} \subset \lambda^{(1)} \subset \lambda^{(2')} \subset \lambda^{(2)} \dots \subset \lambda^{(n)} = \lambda$ , where:

- $\lambda^{(0)}$  is the  $k$ -core of  $\lambda$ .
- Each  $\lambda^{(i)} \setminus \lambda^{(i')}$  is a (possibly empty) horizontal  $k$ -strip.
- Each  $\lambda^{(i')} \setminus \lambda^{(i-1)}$  is a (possibly empty) vertical  $k$ -strip.

We number the ribbons on the strip  $\lambda^{(i)} \setminus \lambda^{(i')}$  by  $i$  and the ribbons forming the strip  $\lambda^{(i')} \setminus \lambda^{(i-1)}$  by  $i'$  for each  $i = 1, 2, \dots, n$ .

FIGURE 10. Semi-standard fillings of the given shape with numbers  $\leq 2$ 

**Definition 3.12.** For a shifted shape  $\lambda$ , we define its  $k$ -ribbon  $Q$ -function and  $P$ -functions as follows:

$$RQ_{\lambda}^{(k)}(X) = \sum_{S \in SSShT^{(k)}(\lambda)} X^{|S|}, \quad RP_{\lambda}^{(k)}(X) = \sum_{S \in SSShT^{*(k)}(\lambda)} X^{|S|}$$

where  $SSShT^{(k)}(\lambda)$  is the set of semi-standard shifted  $k$ -ribbon tableaux of shape  $\lambda$ , and  $SSShT^{*(k)}(\lambda)$  is its subset consisting of tableaux with no marked entries on the ribbons that have boxes on the main diagonal.

The example illustrated in Figure 10 restricted to two variables gives the  $Q$  and  $P$  ribbon functions:

$$\begin{aligned} RQ_{(5,4,2,1)}^{(3)}(x_1, x_2) &= 4x_1^3x_2 + 8x_1^2x_2^2 + 4x_1x_2^3 = Q_{3,1}(x_1, x_2) \\ RP_{(5,4,2,1)}^{(3)}(x_1, x_2) &= x_1^3x_2 + 2x_1^2x_2^2 + x_1x_2^3 = P_{3,1}(x_1, x_2) \end{aligned}$$

Note that in this example, the ribbon  $Q$  and  $P$  functions are Schur  $Q$  and  $P$  positive respectively. One of our main results in this paper will be to show that the positivity holds true in general, and give a formula for the ribbon  $Q$  functions in terms of Schur  $Q$  functions.

**Proposition 3.13.** All the  $k$ -ribbon tableaux of shape  $\lambda$  have the same number  $d$  of ribbons that have a box on the main diagonal. Consequentially, the  $Q$  and  $P$  ribbon functions for  $\lambda$  are related by a scalar:

$$RQ_{\lambda}^{(k)}(X) = 2^d RP_{\lambda}^{(k)}(X)$$

*Proof.* By Corollary 3.7, the ribbons that have a box on the main diagonal correspond to the moves on the abacus where beads are removed. The total number is independent of the order of the moves. In fact, if we denote the number of beads on runner  $i$  by  $|a_i|$  then:

$$d = |a_k| + \sum_{i < (k/2)} \max\{|a_i|, |a_{k-i}|\}$$

□



This proposition will follow from our combinatorial constructions in the following chapters, as well as the main result in this paper:

**Theorem 3.14.** *The  $Q$ -ribbon functions have the following expansion in terms of Schur's  $Q$ -functions.*

$$RQ_{\lambda}^{(k)}(X) = Q_{a_k}(X) \prod_{i \leq \lfloor k/2 \rfloor} Q_{\mu_i}(X)$$

where  $a_k$  is a shifted shape, and the  $\mu_i$  are the unshifted shapes (viewed as skew shifted) described by the  $k$ -quotient of  $\lambda$ . This formula, by Theorem 2.2[11] implies that the  $Q$ -ribbon functions expand positively into Schur's  $Q$  functions.

An equivalent way of stating the formula is, seeing the  $\mu_i$  directly of unshifted shapes

#### 4. FOLDED TABLEAUX

In this section, we will introduce an operation combining two shifted shapes to get an unshifted shape with a specialized diagonal, which we will call a folded diagram. We will later use the folded diagrams, along with their corresponding tableaux in describing the  $k$ -quotient of a shifted shape. We will use the notation  $\delta_n$  to denote the staircase partition  $(n, n-1, \dots, 1)$  of size  $n$ .

**Definition 4.1.** A folded diagram  $\Gamma = (\gamma, \mathfrak{d})$  is an unshifted diagram  $\gamma$  called the underlying shape of  $\Gamma$  along with a specialized main diagonal  $\mathfrak{d}$ , that may or may not be on the shape.

**Definition 4.2.** Let us say we have a pair of shifted shapes (or equivalently strict partitions)  $\alpha$  and  $\beta$ . Denote  $n = \min\{ht(\alpha), ht(\beta)\}$ . Their combination, which will be denoted by  $\alpha \diamond \beta$  will be the folded diagram we obtain by:

- Step I: If one of the shapes has height  $m$  larger than  $n$ , delete its  $n - m$  leftmost columns, so that both shapes have the same number of boxes in their main diagonals.
- Step II: Transpose  $\alpha$ .
- Step III: Paste the two diagrams together along their main diagonals, and calling this the specialized main diagonal  $\mathfrak{d}$ .

**Example:**

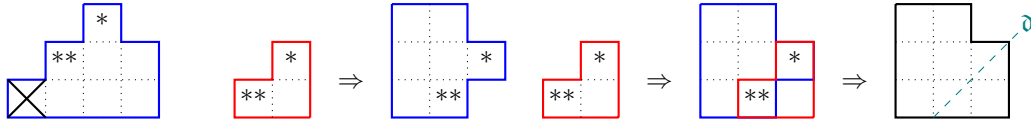


FIGURE 11. The combination of  $\alpha = (4, 3, 1)$  and  $\beta = (2, 1)$

**Proposition 4.3.** Any given folded diagram  $(\gamma, \mathfrak{d})$  can be uniquely described as a combination of two shifted shapes.

*Proof.* Let us denote the difference  $\lambda \setminus \delta_{height(\lambda)}$  by  $\lambda'$ . If  $\mathfrak{d}$  is on the shape, going through  $t \geq 0$  boxes, then  $(\gamma, \mathfrak{d}) = \alpha \diamond \beta$  where  $\alpha = \delta_{k+t} + (\{\gamma_{t+1}, \gamma_{t+2}, \dots, \gamma_n\})'$  and  $\beta = \delta_t + \{\gamma'_{k+t+1}, \gamma'_{k+t+2}, \dots, \gamma'_n\}$ .

If  $\mathfrak{d}$  is  $k \geq 0$  units to the right of the bottom left corner and outside the shape (the case  $k < 0$  is symmetrical) we have  $(\gamma, \mathfrak{d}) = \alpha \diamond \beta$  where  $\alpha = \delta_k + (\gamma)'$  and  $\beta$  is empty.  $\square$

**Definition 4.4.** A standard folded tableau of shape  $\Gamma = (\gamma, \mathfrak{d})$  is a filling of the boxes using numbers  $1, 2, \dots, n$  each exactly once, with numbers increasing left to right and bottom to top.

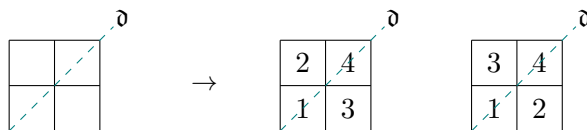


FIGURE 12. Standard Folded Tableaux of shape  $\Gamma = ((2, 2), \mathfrak{d})$

**Definition 4.5.** A semi-standard folded tableau of shape  $\Gamma = (\gamma, \mathfrak{d})$  is a semi-standard filling of the skew-shifted diagram  $\gamma$  with the rules inverted strictly above the specialized diagonal (above  $\mathfrak{d}$  we can have no two  $i$ 's on the same row, and no two  $i$ 's on the same column).

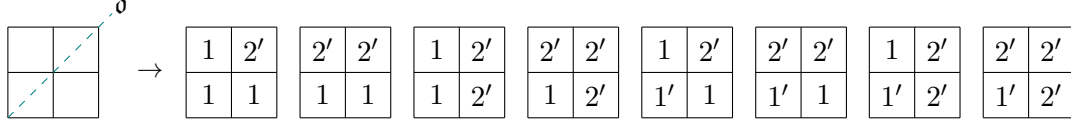


FIGURE 13. Semi-standard folded tableaux of shape  $\Gamma = ((2, 2), \mathfrak{d})$ , using numbers  $\leq 2'$

**Definition 4.6.** We define the folded  $P$  and  $Q$  functions as follows.

$$Q_{(\gamma, \mathfrak{d})}^f(X) = \sum_{S \in \text{SSFT}(\gamma, \mathfrak{d})} X^{|S|}$$

$$P_{(\gamma, \mathfrak{d})}^f(X) = 2^{-\text{size}(\mathfrak{d})} Q_{(\gamma, \mathfrak{d})}^f(X)$$

where  $\text{SSFT}(\gamma, \mathfrak{d})$  denotes the set of all semi-standard folded tableaux of shape  $\Gamma = (\gamma, \mathfrak{d})$ .

The folded shape  $((2, 2), \mathfrak{d})$  illustrated in Figures 4 and 13 has the following folded  $P$  and  $Q$  functions:

$$P_{((2, 2), \mathfrak{d})}^f(X) = m_{31}(X) + 2m_{22}(X) + 4m_{211}(X) + 8m_{1111}(X) = s_{31}(X) + s_{22}(X) + s_{2111}(X) = P_{(31)}(X)$$

$$Q_{((2, 2), \mathfrak{d})}^f(X) = 2^{ht(\mathfrak{d})} P_{((2, 2), \mathfrak{d})}^f(X) = 4(s_{31}(X) + s_{22}(X) + s_{2111}(X)) = Q_{(31)}(X)$$

**Theorem 4.7.** The function  $Q_{(\gamma, \mathfrak{d})}^f(X)$  is independent of the choice of the main diagonal  $\mathfrak{d}$ .

*Proof.* We will prove this by giving a bijection between  $\text{SSFT}(\gamma, \mathfrak{d})$  with semi-standard fillings of shape  $\gamma$  as a skew-shifted tableaux which preserves the number of  $i$ s and  $i$ 's for each  $i$ .

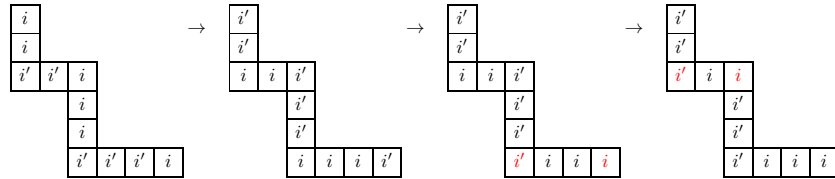
Let  $S \in \text{SSFT}(\gamma, \mathfrak{d})$ . First, for all the boxes above the main diagonal, including the main diagonal, we invert the markings (making each marked number unmarked, and each unmarked number marked). As we can add or remove marks from the main diagonal without causing problems, we will not have to change any markings strictly below main diagonal.

Above the main diagonal, each connected piece containing only  $i$  or  $i$ 's is necessarily a ribbon, as our rules stop us from filling a  $2 \times 2$  box with  $i$ s and  $i$ 's.

With the new markings, we have at most one  $i'$  number on each row which is the rightmost box, and at most one  $i$  on each column being the lowest box.

We will go from bottom the top in the ribbon, and carry the  $i'$  to the leftmost box, or the  $i$  to the highest box, fixing the orders.

When we fix all ribbons above the main diagonal for all  $i$ , then we are left with a semi-standard filling of  $\gamma$  as a skew-shifted tableau.



The process can be inverted by inverting the markings above the main diagonal again, and this time working our way from top to bottom on each ribbon of  $i$ , carrying the at most one  $i'$  to the lowest box on vertical parts and the at most one  $i$  to the rightmost box on horizontal parts.  $\square$

Note that, as the  $Q$  function for a folded shape  $\Gamma = (\gamma, \mathfrak{d})$  is independent of the choice of the main diagonal, we can simply carry  $\mathfrak{d}$  outside the shape, so that we will be looking at the semi-standard fillings of  $\gamma$  as a skew-shifted shape.

**Corollary 4.8.** *The folded  $Q$  functions are Schur  $Q$ -positive. In fact*

$$Q_{\Gamma}^f(X) = Q_{\lambda/\delta_n}(X) = \sum_{\epsilon} f_{\epsilon, \delta_n}^{\lambda} Q_{\epsilon}(X)$$

where  $\lambda$  is a shifted shape with  $\lambda/\delta_n = \gamma$  and  $f_{\epsilon, \delta_n}^{\lambda}$  are the non-negative integers defined by:

$$P_{\epsilon} P_{\delta_n} = \sum_{\lambda} f_{\epsilon, \delta_n}^{\lambda} P_{\lambda}$$

As the folded  $P$  function depends on the size of the main diagonal, it is not independent of  $\mathfrak{d}$ . Instead, it tells us that  $Q_{\gamma}^f(X)$  is divisible by  $2^d$ , where  $d$  is the size of the longest diagonal on  $\gamma$ .

**Corollary 4.9.** *An unshifted shape and its conjugate have the same folded  $Q$ -function.*

*Proof.* For an unshifted shape  $\gamma$ , the conjugation operation gives a bijection of folded tableaux  $(\gamma, \mathfrak{d})$  with  $\mathfrak{d}$  above the shape and folded tableaux  $(\gamma^T, \mathfrak{d}')$  with  $\mathfrak{d}'$  below the shape. As the folded  $Q$ -function is independent of the placement of the specialized diagonal, we have:  $Q_{\gamma}^f(X) = Q_{\gamma^T}^f(X)$   $\square$

## 5. QUOTIENTS OF RIBBON TABLEAUX

On this section, we will introduce the  $k$ -quotient for a shifted diagram, and we will give a bijection between the  $k$ -ribbon tableaux and the fillings of its  $k$ -quotient. Our definition of the  $k$ -quotient matches the one given by Morris and Yaseen in [8], but it also includes information about the diagonals of the quotient through the use of folded tableaux.

**Definition 5.1.** *The  $k$ -quotient of a shifted shape  $\lambda$  with  $k$ -abacus representation  $(\alpha^{(1)}, \alpha^{(2)} \dots \alpha^{(k)})$  will consist of a  $\lfloor k/2 \rfloor$  folded shapes and one shifted shape, defined as follows:*

$$\overline{\Phi}^k(\lambda) = \begin{cases} (\alpha^{(1)} \diamond \alpha^{(k-1)}, \alpha^{(2)} \diamond \alpha^{(k-2)}, \dots, \alpha^{((k-1)/2)} \diamond \alpha^{((k+1)/2)}, \alpha^{(k)}) & k \text{ odd} \\ (\alpha^{(1)} \diamond \alpha^{(k-1)}, \alpha^{(2)} \diamond \alpha^{(k-2)}, \dots, \alpha^{(k/2-1)} \diamond \alpha^{(k/2+1)}, \alpha^{(k/2)} \diamond \emptyset, \alpha^{(k)}) & k \text{ even} \end{cases}$$

The strict partition  $\lambda = \{16, 11, 10, 9, 8, 7, 4, 3, 1\}$  has the 5-quotient:

$$\overline{\Phi}^5(\lambda) = \{(4, 3, 1) \diamond (2, 1), (2) \diamond (2, 1), (2)\} = \{((3, 3, 2), \mathfrak{d}_1), ((3), \mathfrak{d}_2), \{2\}\}$$

where  $\mathfrak{d}_1$  and  $\mathfrak{d}_2$  are the specialized diagonals given by the combination operation.

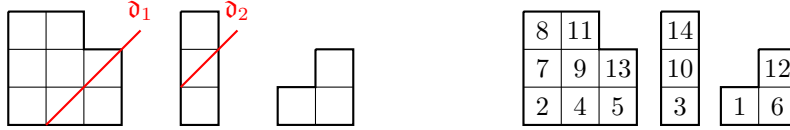


FIGURE 14. The 5-quotient of  $\lambda = (16, 11, 10, 9, 8, 7, 4, 3, 1)$  and a standard filling.

We will call a simultaneous semi-standard filling of the  $\lfloor k/2 \rfloor$  folded shapes and the shifted shape a *semi-standard filling* of the  $k$ -quotient. If this filling uses each number from 1 to  $n$  exactly once, unmarked, it will be called a *standard filling*.

**Theorem 5.2.** *There is a bijection  $\Phi_{\lambda}^k$  between standard  $k$ -ribbon tableaux of shape  $\lambda$  and standard fillings of its  $k$ -quotient such that two ribbons that have the same diagonal value will be mapped to the same shape and diagonal in the quotient.*

*Proof.* Consider a  $k$ -ribbon tableaux  $T$  of shape  $\lambda$  with abacus representation  $(\alpha^{(1)}, \alpha^{(2)} \dots \alpha^{(k)})$ . As each ribbon corresponds to a move in the abacus representation of  $\lambda$ ,  $T$  uniquely corresponds to a sequence of moves from  $(\alpha^{(1)}, \alpha^{(2)} \dots \alpha^{(k)})$  to the  $k$ -core of  $\lambda$ . As we can move independently on each runner pair  $(a_i, a_{k-i})$  and on  $a_k$ , it will suffice to match the moves on  $a_k$  moves to shifted tableaux of shape  $\alpha^{(k)}$ , and the moves on runner pairs  $(a_i, a_{k-i})$  to moves on  $\alpha^{(i)} \diamond \alpha^{(k-i)}$  for each  $i$ .

**Claim:** There is a bijection between sequences of moves from  $a_k$  to the empty runner and standard shifted tableaux of shape  $\alpha^{(k)}$ , where a move from row  $d$  to  $d-1$  on the abacus will correspond to a box on diagonal  $d$ .

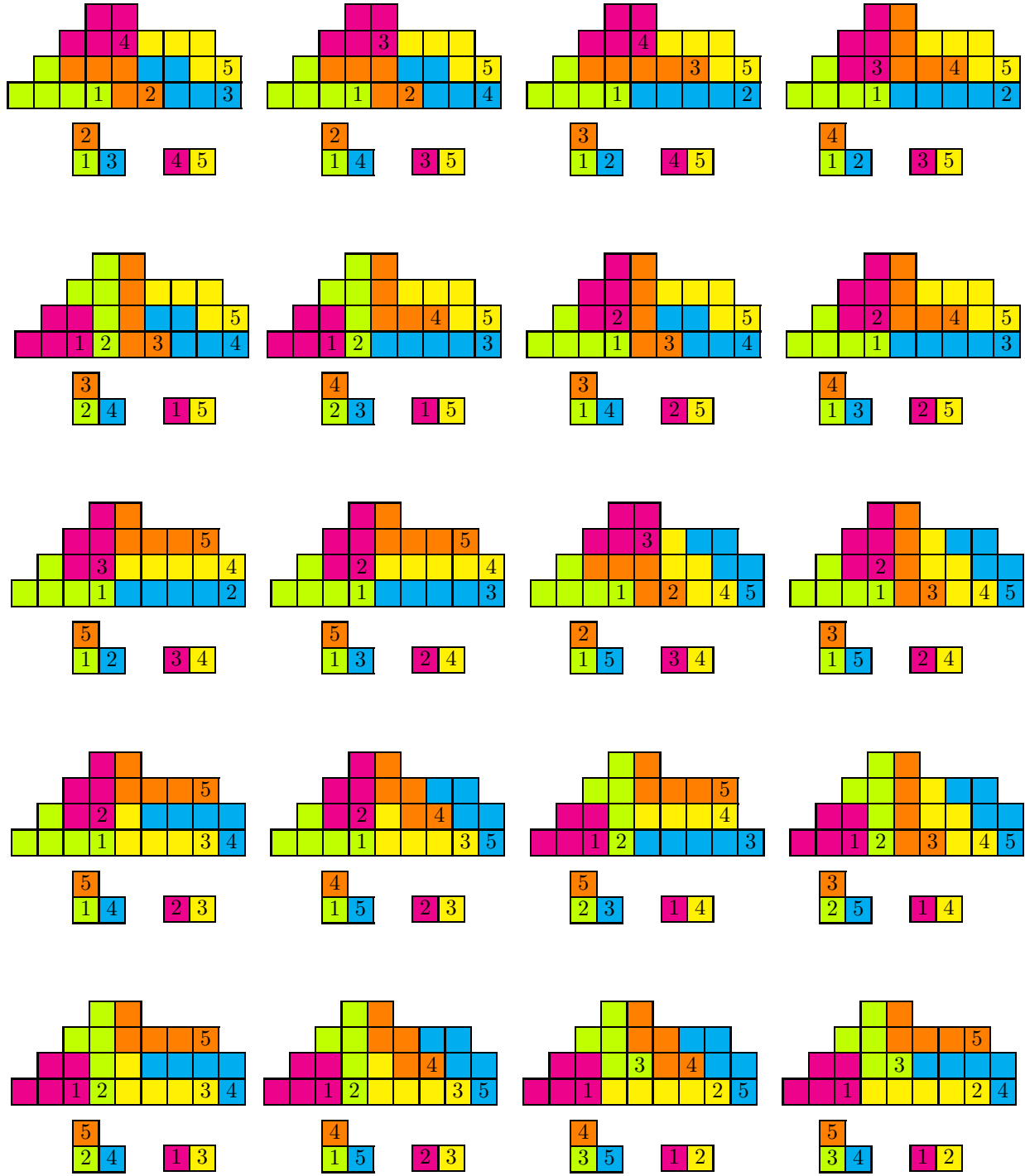


FIGURE 15. Correspondence between the standard 5-ribbon tableaux of shape  $(9, 8, 6, 2)$  with and the standard fillings of its quotient

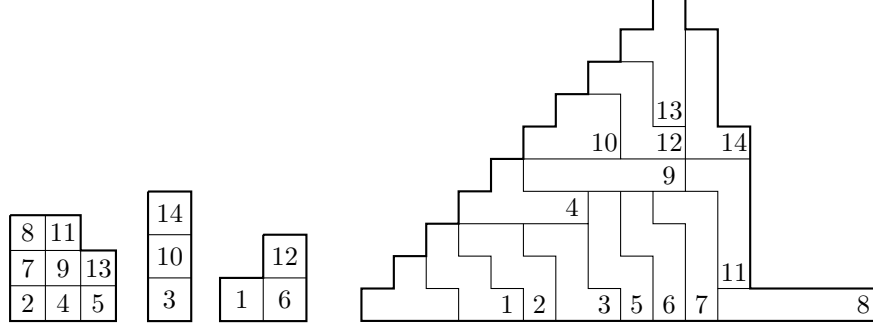


FIGURE 16. A 5-ribbon tableau of shape  $\lambda = (16, 11, 10, 9, 8, 7, 4, 3, 1)$  with the corresponding filling of the 5-quotient

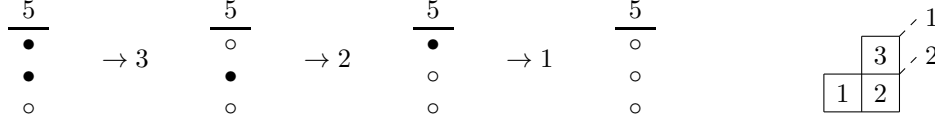


FIGURE 17. Moves on runner  $a_5$  can be matched to a standard filling of the shifted diagram  $\alpha^{(5)} = (2, 1)$

Proof: A bead on position on the  $i$ th row on runner  $a_k$  will make a total of  $i$  moves,  $i - 1$  to one row higher, and one last move to be removed. We map these moves to a row of  $i$  boxes so that the move from position  $j$  to  $j - 1$  will correspond to a box on diagonal  $j$ , and the removal move will correspond a box on the main diagonal. This means if  $a_k$  has beads on positions  $i_1 > i_2 > \dots > i_t$  we will map them to the shifted diagram  $\alpha^{(k)} = (i_1, i_2, \dots, i_t)$ . The only conditions on these moves are, before moving to row  $j - 1 < i$  a bead must move to the row  $j$  and the bead above on  $a_k$  must move to  $j - 2$ . Assume we number the moves in decreasing order with numbers from 1 to  $|\alpha^{(k)}|$ . This will give us a filling of  $\alpha^{(k)}$ , where the conditions exactly correspond to the tableaux conditions: numbers need to increase along columns and rows. Now, we can turn our attention to runner pairs  $a_i, a_{k-i}$ .

**Claim:** There is a bijection between sequences of moves from the pair of runners  $(a_i, a_{k-i})$  to the abacus core and standard unshifted tableau of shape  $\alpha^{(i)} \diamond \alpha^{(k-i)}$ , where moves removing beads are mapped to the specialized diagonal of  $\alpha^{(i)} \diamond \alpha^{(k-i)}$ , and moves on runner  $a_i$  (resp.  $a_{k-i}$  from row  $d$  to  $d - 1$  are mapped to the diagonal  $d$  units to the left (resp. right).

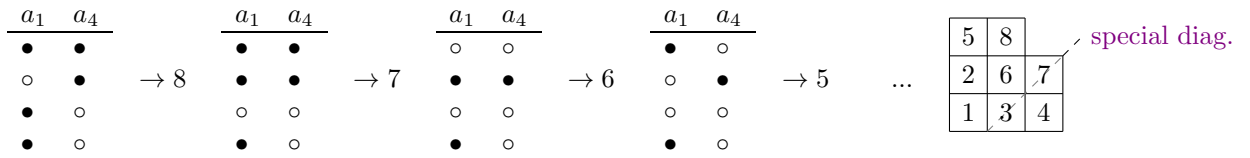


FIGURE 18. Moves on runners  $a_1$  and  $a_4$  give a standard filling of the folded diagram  $\alpha^{(1)} \diamond \alpha^{(1)}$

Proof: The move sequences on each runner can be matched to a shifted tableau of corresponding shape as in Claim 1, except now we have an additional constraint: To remove a bead from the first row one runner, we must simultaneously remove a bead from the first row of the second runner. This implies that the main diagonals of the two shapes must contain the same numbers, and they will be on the main diagonal. Furthermore, if one runner has  $q$  more beads than the other one, these can not be removed, and the moves which depend on the removal of these beads can not be made, which means the subdiagram of shape  $\delta_q$  inside the larger shape will be left empty. When  $k$  is even, we can move ribbons up on runner  $a_{k/2}$  but not remove them, as if it has a conjugate runner with no beads, so any remark we made on  $\alpha^{(i)} \diamond \alpha^{(k-i)}$  above automatically applies to  $\alpha^{(k/2)} \diamond \emptyset$ .  $\square$

The correspondence of diagonals gives us a way of labeling the diagonals of the quotient to match the values of the original shape. This way, the shifted shape  $\alpha^{(k)}$  will have diagonals  $0, k, 2k, \dots$  and the folded shapes  $\alpha^{(i)} \diamond \alpha^{(k-i)}$  will have diagonals:  $\{\dots, i + 2k, i + k, k - i, 2k - i, 3k - i, \dots\}$  where the specialized diagonal  $\mathfrak{d}_i$  will have the diagonal value  $k - i$ . Note that the diagonal values  $i < (k - 1)/2$  do not appear, that is because the main diagonals correspond to double ribbons, and we set the head of double ribbon of size  $(i, k - i)$  to be the head of the larger piece by convention.

**Corollary 5.3.** *A  $k$ -ribbon  $R$  has a box on the main diagonal of  $\lambda$  if and only if  $\text{diag}(R) \leq k$ , and the total number  $d$  of such ribbons is the same for every  $k$ -ribbon tableau of shape  $\lambda$ .*

**Definition 5.4** (Standardization). *Consider a semi standard  $k$ -ribbon tableaux  $T$  of shape  $\lambda$ , with  $|\lambda| = n$ . Standardization of  $T$ , denoted  $St(T)$  is the standard  $k$ -ribbon tableaux of shape  $\lambda$  that we obtain by the following numbering:*

- We number the cells in the order  $1' < 1 < 2' < 2 \dots$
- If there is more than one cell of label  $i$ , we order them so that their diagonal values will be increasing.
- If there is more than one cell of label  $i'$ , we order them so that their diagonal values will be decreasing.

**Proposition 5.5.**  *$St(T)$  is well defined.*

*Proof.* Note that we can not have two  $i$ s or  $i'$ s on the same diagonal, because as each number corresponds to a skew shifted shape, the ribbon under one and to the right of the other would also be labeled the same, making the resulting shape neither a horizontal nor a vertical  $k$ -strip. Also,  $St(T)$  is going to be a standard  $k$ -ribbon tableaux, as it is possible to remove the ribbons in the order  $n, n - 1, \dots, 1$ . In particular, ribbons labeled  $i$  form a horizontal strip, and can be removed in the order diagonals are increasing. Ribbons labeled  $i'$  form a vertical strip and can be removed in the order their diagonals are decreasing.  $\square$

As the diagonal values of the shape  $\lambda$  can be carried to the quotient shape, the standardization map can also be applied to a semi-standard filling of the quotient, to obtain a standard filling.

**Theorem 5.6.** *We can extend  $\Phi_\lambda^k$  to a bijection between semi-standard  $k$ -ribbon tableaux of shape  $\lambda$ , and semi-standard fillings of its  $k$ -quotient that preserves the total number of times  $i$  or  $i'$  occurs for each  $i$ .*

*Proof.* Let  $T$  be a semi-standard  $k$ -ribbon tableaux of shape  $\lambda$ , given by the sequence  $\lambda_0 \subset \lambda_{1'} \subset \lambda_1 \subset \lambda_{2'} \subset \lambda_2 \dots \subset \lambda_t = \lambda$  of shifted diagrams. As our definition of standardization respects the inclusion order,  $St(T)$  restricted to any  $\lambda_i$  gives a standardization of  $\lambda_i$ . The same is true for  $\lambda_{i'}$ s. Let us apply the  $\Phi_\lambda^k$  to the standardization of  $T$ . This gives us a bijection  $\phi$  between ribbons of  $T$  and the boxes on the  $k$ -quotient. This also can be restricted to the subdiagrams  $\lambda_i$  and  $\lambda_{i'}$ , giving a sequence  $\Phi^k(St(\lambda_0)) \subset \Phi^k(St(\lambda_{1'})) \subset \Phi^k(St(\lambda_1)) \dots \subset \Phi^k(\lambda_t) = \Phi^k(\lambda)$ . Here, the subset relation is defined pointwise in the  $(k + 1)$ -tuples of quotient diagrams.

**Claim:** The filling of the  $k$ -quotient obtained by this is a semi-standard filling, and is equal to  $\Phi_\lambda^k(T)$  if the filling  $T$  is standard.

*Proof:* The second part of the claim is obvious. For the first part, we need to show that the filling of each  $a_i \diamond a_{k-i}$  gives a semi-standard folded shape and  $a_k$  gives a semi-standard shifted shape. Let us look first at the case of  $a_k$ . To obtain a contradiction, let us assume there are two boxes  $B_1$  and  $B_2$  on  $a_k$  that are marked  $j$  and are on the same column (so that they do not form a horizontal 1-strip). Without loss of generality, we can take  $B_2$  to be the higher one. This means if we name the corresponding ribbons on  $\lambda$  respectively  $R_1$  and  $R_2$ , we have  $\text{diag}(R_2) < \text{diag}(R_1)$ . Also, as they both are labeled  $j$ , they are on a horizontal  $k$ -strip, specifically  $H(R_2)$  lies strictly to the right of  $H(R_1)$ . These together imply that  $H(R_2)$  must also be strictly above  $H(R_1)$ . Remember that the cells labeled  $j$  form a skew shape, so the cell  $C$  that is on the same row as  $H(R_1)$  and the same column as  $H(R_2)$  must also be in  $\lambda_i$  with its diagonal value higher than those of  $H(R_1)$  and  $H(R_2)$ . This implies it is not on  $R_1$  or  $R_2$ . It must be on a different ribbon  $R_3$  on the horizontal  $k$ -strip. As  $R_2$  and  $R_3$  have boxes in the same column with the box of  $R_2$  above, we can not remove  $R_3$  before  $R_2$ . This implies  $H(R_3)$  must be strictly to the right of  $H(R_2)$  and consequently to the right of  $C$ . This can not happen as  $C$  is on  $R_3$ , by 3.4. Symmetrically, no two cells marked  $j'$  can be on the same row, so we indeed have a semi-standard filling of  $a_k$ . Now consider the boxes marked  $j$  on  $a_i \diamond a_{k-i}$  in some  $i$ . As they come from the difference  $\Phi^k(St(\lambda_j)) \setminus \Phi^k(St(\lambda_{j'}))$ , they form a skew shape. Also, the boxes that are labeled  $j$  to the right of the main diagonal form a horizontal strip by the same reasoning in the case of  $\lambda$ .

The boxes labeled  $j$  to the left of the main diagonal form a vertical strip, as we have the inverted version of the same rules. The  $j'$  case is again symmetrical.

Now let us define the inverse of this operation. Given a semi-standard filling  $\bar{T}$  of the  $k$ -quotient, as the  $k$ -quotient has the diagonal values induced by  $\lambda$ , we can apply the same standardization algorithm to the quotient, to get a standard filling  $St(\bar{T})$  of the quotient. Applying  $\Phi_\lambda^{k-1}$  to this filling gives a standard filling of  $\lambda$ . We can use this bijection between cells of the quotient and ribbons of  $\lambda$  to carry the labels in  $\bar{T}$  to the corresponding ribbons in  $\lambda$ . Note that, this inverts the above operation by definition.

**Claim:** The inverse operation takes  $\bar{T}$  to a semi-standard filling of  $\lambda$ .

Proof: Let  $R$  and  $S$  be two ribbons marked  $j$  on  $\lambda$ . We will show that they form an horizontal strip. The case of  $j'$  is symmetrical. First note that we can not have  $diag(R) = diag(S)$ , as that would imply the corresponding cells in the quotient are both in the same  $a_i \diamond a_{k-i}$  (or both in  $a_k$ ) on the same diagonal, which is not possible. Let us assume, without loss of generality, that  $diag(R) > diag(S)$ . Then, in the standardization, the label of  $R$  will be higher than the label of  $S$ , meaning  $R$  can be removed before  $S$ :  $H(R)$  can not be below  $H(S)$  in the same column. We will show that, in this case we have  $H(R)$  strictly to the right of  $H(S)$  implying they form a horizontal strip. Assume otherwise. Then, by the diagonal relationship,  $H(R)$  must be strictly below  $H(S)$ , in a row strictly to the left. Then we can look at the ribbon containing the cell  $C$  in the same row as  $H(R)$  and the same column as  $H(S)$ . As in the proof of the first claim, there is no way to label this ribbon, giving us a contradiction.  $\square$

This bijective relationship shows that the  $k$ -ribbon  $Q$  function is equal to the product of the folded  $Q$  functions of the folded shapes and the Schur  $Q$  function of the shifted shape in its diagonal, proving our theorem:

$$RQ_\lambda^{(k)}(X) = Q_{a_k}(X) \left( \prod_{i < k/2} Q_{\mu_i}(X) \right) \left( \mathbb{1}_{2\mathbb{Z}}(k) Q_{a_{k/2} \diamond \emptyset}^f(X) \right) = Q_{a_k}(X) \prod_{i \leq k/2} Q_{\mu_i}(X)$$

where  $\mathbb{1}_{2\mathbb{Z}}(k)$  is the indicator function and  $\mu_i$  is the underlying skew-unshifted shape of  $a_i \diamond a_{k-i}$  if  $i < k/2$  and  $a_{k/2} \diamond \emptyset$  if  $i = k/2$ .

Note that the Schur  $Q$  functions are  $Q$  ribbon functions themselves, as  $k\lambda = (k\lambda_1, k\lambda_2 \dots k\lambda_n)$  has  $RQ_{k\lambda}^{(k)}(X) = Q_\lambda$ .

## 6. PEAK FUNCTIONS OF RIBBON TABLEAUX

The reading word of a  $k$ -ribbon tableau is a listing of the numbers on the heads of the ribbons, read left to right, top to bottom.

**Definition 6.1.** A marked standard shifted  $k$ -ribbon tableau  $T'$  of shape  $\lambda$  is defined to be a standard shifted  $k$ -ribbon tableau  $T$  of shape  $\lambda$  together with a subset  $M$  of  $[n]$  determining the marked coordinates. On the Young diagram, for all  $i$  in  $M$ , we replace the label of  $R_i$  with  $i'$ . We will denote the set of all the marked versions of  $T$  by the set  $Mark(T)$ .

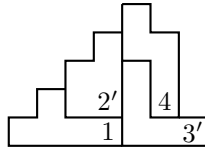


FIGURE 19. This marked 5-ribbon tableau has reading word  $2', 4, 1, 3'$

**Theorem 6.2.** The  $k$ -ribbon  $Q$ -function of a shifted shape  $\lambda$  we have can be written in terms of descent functions and peak functions as follows:

$$RQ_\lambda^{(k)}(X) = \sum_{T' \in SShT^{\pm(k)}(\lambda)} F_{Des(T')} \quad RQ_\lambda^{(k)}(X) = \sum_{T \in SShT^{(k)}(\lambda)} 2^{|Peak(Des(T))|+1} G_{Peak(Des(T))}$$

where  $SShT^{(k)}(\lambda)$  is the set of marked standard shifted  $k$ -ribbon tableaux of shape  $\lambda$ , and  $SShT^{\pm(k)}(\lambda)$  is the set of such tableaux with no marked numbers on ribbons with boxes on the main diagonal.

To prove this theorem, we need to define runs. For a subset  $D$  of  $[n]$  the number of runs of  $D$  as the minimum number  $t$  where  $D = D_1 \cup D_2 \dots \cup D_t$  and each  $D_i$  consists of consecutive numbers. Each subset  $D_i$  is a *run* of  $D$ .

For example, if  $D = \{2, 3, 5, 8, 9, 10\} = \{2, 3\} \cup \{5\} \cup \{8, 9, 10\}$ , then  $\text{run}(D) = 3$ . Note that  $\text{Peak}(D) = \{2, 5, 8\}$  is formed by the first elements of each  $D_i$ .

**Proposition 6.3.** *We can calculate the number of the peaks of a tableaux  $T$  from its descent set as follows:*

$$|P| = \begin{cases} \text{run}(D) & 1 \notin \text{Des}(T) \\ \text{run}(D) - 1 & 1 \in \text{Des}(T) \end{cases}$$

*Proof.* This comes from the fact that the elements  $j$  of the peak set satisfy  $j \in D$  and  $j - 1 \notin D$  for all  $j > 1$ , so that elements of the peak set are given by the smallest elements of each run, excepting the case if there is a run starting with 1.  $\square$

**Lemma 6.4.** *For any  $T' \in \text{Mark}(T)$ , the descent set of  $T'$  is independent of whether  $i$  is marked if and only if:*

- $i > 1$  with  $i - 1 \in \text{Des}(T)$ ,  $i \notin \text{Des}(T)$  or
- $i = 1 \notin \text{Des}(T)$

*The number of such  $i$  is given by  $|\text{Peak}(T)| - 1$ .*

*Proof.* The first part comes from Lemma 2.1. Also, a number  $i$  satisfies these conditions iff  $i$  is one lower than the lowest number of a run of  $\text{Des}(T)$ . That means, if  $1 \notin D$ , it will be equal to the number of runs of  $\text{Des}(T)$ . If  $1 \in T$ , it will be equal to the number of runs  $-1$ .  $\square$

**Proposition 6.5.** *For any  $T' \in \text{Mark}(T)$ , we have :*

$$\text{Spike}(T') \supset \text{Peak}(T)$$

*Proof.* For all  $i \in \text{Peak}(\text{Des}(T))$  we have  $i \in \text{Des}(T)$  and  $i - 1 \notin \text{Des}(T)$ . For any given  $T'$  if  $i$  is unmarked on  $T'$ , then  $i \in \text{Des}(T')$  and  $i - 1 \notin \text{Des}(T')$  so  $i \in \text{Spike}(T')$ . Otherwise  $i$  is marked, so that we have  $i \notin \text{Des}(T')$  and  $i - 1 \in \text{Des}(T')$ , implying again that  $i \in \text{Spike}(T')$ .  $\square$

The proposition above shows that the descent map takes the elements of  $\text{Mark}(T)$  to subsets  $D$  of  $[n - 1]$  with  $\text{Spike}(D) \subset \text{Peak}(T)$ . Next, we will show that this map is surjective. In fact, we will prove the stronger statement that the preimage of every element is of the same size.

**Lemma 6.6.** *Assume  $D$  is a subset of  $[n - 1]$  satisfying  $\text{Spike}(D) \supset \text{Peak}(T)$ . Then, there is a marked version  $T'$  of  $T$  such that  $\text{Des}(T') = D$ .*

*Proof.* Let us generate a marked version  $T'$  of  $T$  as follows: Starting with  $i = 1$ , at Step  $i$  we mark  $i$  if  $i \in \text{Des}(T)$  and  $i \notin D$ , and we mark  $i + 1$  if  $i \notin \text{Des}(T)$  and  $i \in D$  (marking the same number a second time has no effect). Then we move on to the next number, till we go through all  $i \leq n - 1$ .

Let us verify that the descent set of  $T'$  is indeed equal to  $D$ . For a fixed  $i$  assume  $i \in \text{Des}(T)$ . Then by Lemma 2.1  $i$  is a descent of  $T'$  iff  $i$  is unmarked. Therefore, it is sufficient to show  $i$  is unmarked iff  $i \in D$ . If  $i \notin D$ , then we marked  $i$  on Step  $i$ , so  $i \in \text{Des}(T')$ . Otherwise  $i \in D$ , and we can only have marked  $i$  at step  $i - 1$ . This implies  $i - 1 \notin \text{Des}(T)$  and  $i - 1 \in D$ . This contradicts our assumption  $\text{Spike}(D) \supset \text{Peak}(T)$  as  $i$  is a peak of  $T$  but not a spike of  $T$ . The case  $i \notin \text{Des}(T)$  is similar.  $\square$

**Proposition 6.7.** *The descent map taking elements of  $\text{Mark}(T)$  to subsets  $D$  of  $[n - 1]$  with  $\text{Spike}(D) \supset \text{Peak}(\text{Des}(T))$  is a  $2^m$  to one cover, where  $m = |\text{Peak}(\text{Des}(T))| + 1$*

*Proof.* The number of subsets  $D$  of  $[n - 1]$  with  $\text{Spike}(D) \supset \text{Peak}(\text{Des}(T))$  is given by  $2^{n-1-|\text{Peak}(\text{Des}(T))|}$ . By Lemma 6.6, we know that the descent map is surjective. By Lemma 6.4, the preimage of each element under the descent map contains at least  $2^m$  elements.  $2^m \times 2^{n-1-|\text{Peak}(\text{Des}(T))|} = 2^n$  which is the cardinality of  $\text{Mark}(T)$ , so the preimage of each element must contain exactly  $2^m$  elements.  $\square$

Now we are ready to prove Theorem 6.2 from the beginning of the section.



*Proof of Theorem 6.2.* Let  $S$  be a semi-standard  $k$ -ribbon tableaux of shape  $\lambda$ . We have already defined the standardization of  $S$  (Definition 5.4). Let us denote by  $St'(S)$  the marked standardization of  $S$ , which is simply standardization while keeping the marked cells marked. We will show that there is a bijection  $\phi_{T'}$  between semi-standard  $k$ -ribbon tableaux  $S$  that standardize to  $T'$  and the combinations refining  $Des(T')$ , satisfying  $x^{|S|} = x^{\phi_{T'}(S)}$ . This will imply:

$$\sum_{T' \in SShT\pm^{(k)}(\lambda)} F_{Des(T')}(X) = \sum_{T'} \sum_{C \in Ref(Des(T'))} X^C = \sum_{T'} \sum_C x^{|\phi_{T'}^{-1}(C)|} = \sum_{S \in SShT^{(k)}(\lambda)} X^{|S|} = RQ_\lambda^{(k)}(X)$$

where for a combination  $C = (c_1, c_2, \dots, c_t)$  we use  $X^C$  to denote  $x_1^{c_1} x_2^{c_2} \dots x_t^{c_t}$ .

Assume  $S$  has  $St'(S) = T'$ . We define  $\phi_{T'}(S) = (i_1, i_2, \dots)$  where  $i_m$  stands for the total number of  $m$ 's and  $m$ 's on  $S$ . As  $S$  has  $n$  ribbons,  $\phi_{T'}(S)$  will be a combination of  $n$  that satisfies  $x^{|S|} = x^{\phi_{T'}(S)}$ .

**Claim:**  $S$  refines  $Des(T')$ .

*Proof:* Let  $S$  be a semi-standard filling with  $St'(S) = T'$ . Consider the pre-image of ribbon  $R_i$  (the unique ribbon labeled  $i$  or  $i'$  on  $T'$ ) under  $St'$ . We will denote the label of this ribbon in  $S$  by  $St'^{-1}(i)$ . To prove that  $S$  refines  $Des(T')$ , it is sufficient to show that if  $i$  is a descent of  $T'$ , then  $St'^{-1}(i)$  and  $St'^{-1}(i+1)$  are not both elements of  $\{m, m'\}$  for some  $m$  (Note that, by the standardization algorithm, we will have  $St'^{-1}(i) \leq St'^{-1}(i+1)$  in any case).

Let  $i$  be a descent of  $T'$ . By Lemma 2.1, there are two possibilities:

- Case 1-  $i \in Des(T)$  and  $i$  is not marked in  $T'$ : Then  $St'^{-1}(i)$  is an unmarked number  $m$ .  $St'^{-1}(i) \geq m$ , so it can not be  $m'$ . Assume it is also  $m$ . Then, we have two ribbons labeled  $m$ , but  $diag(R_i) > diag(R_{i+1})$  by the definition of standardization (Definition 5.4).
- Case 2-  $i \notin Des(T)$  and  $i+1$  is marked in  $T'$ : This means  $St'^{-1}(i+1)$  is a marked number  $m'$ , and  $St'^{-1}(i) \geq m'$  can not be  $m$ . It can not be  $m'$  either, because as in the first case, we get two ribbons labeled  $m'$  but  $diag(R_i) < diag(R_{i+1})$ , which as in Case 1, contradicts the definition of standardization.

**Claim:** For any combination  $C$  refining  $Des(T)$  there is a unique  $S$  that standardizes to  $T'$  with  $\phi_{T'}(S) = C$

*Proof:* Let  $C = (c_1, c_2, \dots, c_T)$  be a combination of  $n$  that refines  $Des(T')$ . We will define  $S$  by labeling the ribbons  $R_1$  to  $R_{c_1}$  with 1, ribbons  $R_{c_1+1}$  to  $R_{c_1+c_2}$  with 2,  $R_{c_1+c_2+1}$  to  $R_{c_1+c_2+c_3}$  with 3 and so on, and then marking the image of  $R_i$  iff  $i$  is marked in  $T$ . We need to show that this  $S$  is semi-simple, and it standardizes to  $T'$ . Uniqueness then, comes from the fact that the placement of the markings are preserved.

Assume ribbons  $R_i$  and  $R_{i+1}$  have the same unmarked label  $m$ . Then,  $i \notin Des(T')$  and  $i$  and  $i+1$  are both not labeled in  $T'$ , so we must have  $i \notin Des(T)$  by Lemma 2.1. That means,  $diag(R_i) < diag(R_{i+1})$ , so unmarked numbers are ordered so that their diagonals will increase in  $T'$ . Similarly, if  $R_i$  and  $R_{i+1}$  both have the same marked label  $m'$ , then  $i \in Des(T')$  and  $diag(R_i) > diag(R_{i+1})$ . These mean that we have  $St'(S) = T$ .

Additionally, we can remove ribbon  $R_{i+1}$  before  $R_i$ . This implies, if they are both labeled  $m$ ,  $H(R_{i+1})$  is going to be strictly to the right of  $H(R_i)$  as  $diag(R_i) < diag(R_{i+1})$ . If they are both labeled  $m'$ ,  $H(R_{i+1})$  is going to be strictly above  $H(R_i)$  as  $diag(R_i) > diag(R_{i+1})$ . This proves the expansion of the  $k$ -ribbon function in terms of descent functions. The peak function expansion follows by Proposition 6.7:

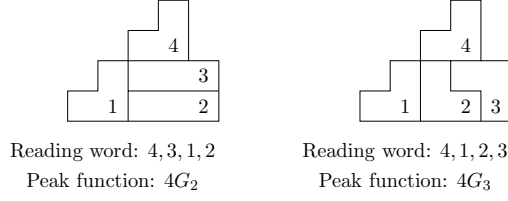
$$RQ_\lambda^{(k)}(X) = \sum_{T' \in SShT\pm^{(k)}(\lambda)} F_{Des(T')} = \sum_T \sum_{T' \in Mark(T)} F_{Des(T')} = \sum_T 2^{|Peak(Des(T))|+1} G_{Peak(Des(T))}$$

□

## 7. DEFINING A Q-ANALOGUE: COUNTER-EXAMPLES

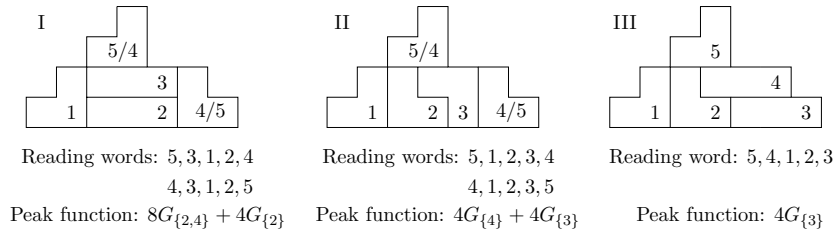
In [5], Lascoux, Leclerc and Thibon give a  $q$ -analogue for the  $k$ -ribbon functions for the unshifted case that is Schur positive. For this, they use the spin statistic on ribbon tableaux which depends on the total height of its ribbons. In this section, we will discuss the difficulty of extending the concept of height to double ribbons in a way that would give a Schur  $Q$ -positive  $q$ -deformation of our shifted  $k$ -ribbon functions.

We will look at two examples where all the different fillings of the same shifted shape need to have the same total height for Schur  $Q$ -positivity. Such examples are helpful in illustrating what properties a non-trivial height definition needs to satisfy.

FIGURE 20. The two 3 ribbon tableaux of shape  $\{5, 4, 2, 1\}$  and their corresponding peak functions

**Theorem 7.1.** *There is no “intrinsic” definition of the height of a double ribbon, which, along with the usual definition of heights for the single ribbon, gives a Schur  $Q$ -positive, or even a Schur positive function. Here by intrinsic, we mean there is no definition that only comes from the shape of the double ribbon, and is independent of its placement or the other ribbons in the shape.*

*Proof.* If we consider the example in Fig 20, we can see that the only difference between the two fillings is the placement of ribbons 2 and 3. For any intrinsic definition, the heights of the double ribbons 4 and 1 would match in the two fillings. The total height of 2 and 3 being higher on the shape on the right, we get a function  $4q^c G_2 + 4q^d G_3$  with  $c \neq d$  which is not Schur  $Q$ -positive. In fact,  $G_2$  and  $G_3$  by themselves are not even Schur positive or symmetric functions.  $\square$

FIGURE 21. Standard 3-ribbon tableaux of shape  $(7, 5, 2, 1)$  and their corresponding peak functions

**Theorem 7.2.** *There is no definition of the height of a double ribbon depending solely on the columns the double ribbon intersects, which, along with the usual definition of heights for the single ribbon, gives a Schur  $Q$ -positive result.*

*Proof.* Consider the 3-ribbon tableaux of shape  $(7, 5, 2, 1)$  illustrated in Figure 21. Assume that we define the height of the single ribbon as in the unshifted case, and we pick a definition for the height of a double ribbon depending on the columns it intersects. Let us denote the cospins of the three fillings on Figure 21 by  $c_1, c_2$  and  $c_3$  from right to left. This gives us the following  $q$  version for the  $k$ -ribbon function:

$$8q^{c_1} G_{\{2,4\}} + 4q^{c_1} G_{\{2\}} + 4q^{c_2} G_{\{4\}} + 4q^{c_2} G_{\{3\}} + 4q^{c_3} G_{\{3\}}$$

Schur's  $Q$ -functions of size 5 are given by:

$$\begin{aligned}
 P_{(5)} &= 2G_{\{\}} \\
 P_{(4,1)} &= 4G_{\{2\}} + 4G_{\{3\}} + 4G_{\{4\}} \\
 P_{(3,2)} &= 4G_{\{3\}} + 8G_{\{2,4\}}
 \end{aligned}$$

So, for Schur  $Q$ -positivity, the terms  $G_{\{2\}}$  and  $G_{\{3\}}$  need to have the same coefficient, meaning  $c_1 = c_2$ , leaving us with:

$$4q^{c_1} P_{(4,1)} + 4(2q^{c_1} G_{\{2,4\}} + q^{c_3} G_{\{3\}})$$

We can see that the only Schur  $Q$ -function with the term  $G_{\{2,4\}}$  is  $P_{(3,2)}$  and to obtain it we also must have  $c_1 = c_3$ . This implies that fillings II and III, which have the same double ribbons have the same total height, which contradicts the fact that the single ribbons 2 and 3 have a higher total height in III.  $\square$

Another interesting point about Figure 21 is that fillings II and III have the same shifted part, so tweaking the definition only the shifted part would not eliminate the problem. Other possible approaches are to alter the definition of the single ribbon, or to avoid the geometric approach and consider how the ribbons interact in the quotient. In pursuing these approaches, the author was not able to arrive at a satisfactory definition, but the possibilities are not yet exhausted and if indeed a non-trivial definition exists, it might involve new and interesting combinatorial constructions.

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## REFERENCES

- [1] Ira M. Gessel. Multipartite  $P$ -partitions and inner products of skew Schur functions. In *Combinatorics and algebra (Boulder, Colo., 1983)*, volume 34 of *Contemp. Math.*, pages 289–317. Amer. Math. Soc., Providence, RI, 1984.
- [2] J. Haglund, M. Haiman, and N. Loehr. A combinatorial formula for Macdonald polynomials. *J. Amer. Math. Soc.*, 18(3):735–761, 2005.
- [3] Gordon James and Adalbert Kerber. *The representation theory of the symmetric group*, volume 16 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1981. With a foreword by P. M. Cohn, With an introduction by Gilbert de B. Robinson.
- [4] Tadeusz Józefiak. Schur  $Q$ -functions and cohomology of isotropic Grassmannians. *Math. Proc. Cambridge Philos. Soc.*, 109(3):471–478, 1991.
- [5] Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon. Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras, and unipotent varieties. *J. Math. Phys.*, 38(2):1041–1068, 1997.
- [6] Bernard Leclerc and Jean-Yves Thibon. Littlewood-Richardson coefficients and Kazhdan-Lusztig polynomials. In *Combinatorial methods in representation theory (Kyoto, 1998)*, volume 28 of *Adv. Stud. Pure Math.*, pages 155–220. Kinokuniya, Tokyo, 2000.
- [7] I. G. Macdonald and and. *Symmetric functions and Hall polynomials*. Oxford Classic Texts in the Physical Sciences. The Clarendon Press, Oxford University Press, New York, second edition, 2015. With contribution by A. V. Zelevinsky and a foreword by Richard Stanley, Reprint of the 2008 paperback edition [MR1354144].
- [8] A. O. Morris and A. K. Yaseen. Some combinatorial results involving shifted Young diagrams. *Math. Proc. Cambridge Philos. Soc.*, 99(1):23–31, 1986.
- [9] Jørn B. Olsson. *Combinatorics and representations of finite groups*, volume 20 of *Vorlesungen aus dem Fachbereich Mathematik der Universität GH Essen [Lecture Notes in Mathematics at the University of Essen]*. Universität Essen, Fachbereich Mathematik, Essen, 1993.
- [10] J. Schur. über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen. *J. Reine Angew. Math.*, 139:155–250, 1911.
- [11] John R. Stembridge. Shifted tableaux and the projective representations of symmetric groups. *Adv. Math.*, 74(1):87–134, 1989.
- [12] John R. Stembridge. Enriched  $P$ -partitions. *Trans. Amer. Math. Soc.*, 349(2):763–788, 1997.

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